

On products of T -ideals in free algebras and free group algebras

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Abstract

Let F be a field and A a free associative F -algebra or a group algebra of a free group with an infinite set X of generators. We find a necessary and sufficient condition for the inclusion $I' \subseteq I$, where $I = I_1 \dots I_k$ and $I' = I'_1 \dots I'_l$ are any products of T -ideals in A . A canonical reformulation in terms of products of group representation varieties answers a question posed in 1986 [O].

1 Introduction

Let $F\langle X \rangle = F\langle x_1, x_2, \dots \rangle$ be a free associative algebra over a field F with 1 and with a countable set of free non-commuting generators $X = \{x_1, x_2, \dots\}$. Recall that an ideal I of $F\langle X \rangle$ is called a T -ideal if $\mu(I) \subseteq I$ for every endomorphism μ of $F\langle X \rangle$. As usual, a product IJ of ideals I and J is the set of finite sums $\sum a_i b_i$ where $a_i \in I, b_i \in J$. We prove

Theorem 1.1. *Let V and V' be products of T -ideals in $F\langle X \rangle$: $V = V_1 \dots V_k$, and $V' = V'_1 \dots V'_l$. Then we have $V' \subseteq V$ if and only if there exists a T -ideal I of $F\langle X \rangle$ with two T -ideal factorizations*

$$I = I_1 \dots I_k = I'_1 \dots I'_l$$

such that

$$I_1 \subseteq V_1, \dots, I_k \subseteq V_k \text{ and } V'_1 \subseteq I'_1, \dots, V'_l \subseteq I'_l.$$

The elements of $F\langle X \rangle$ are called (non-commutative) *polynomials* in this paper. Recall that an F -algebra R satisfies an identity $f(x_1, \dots, x_n) = 0$ for a polynomial $f(x_1, \dots, x_n) = f$ if f vanishes under every homomorphism $F\langle X \rangle \rightarrow R$. A *variety* of associative algebras is a class of algebras consisting of all associative algebras with 1 that satisfy some set of identities. For every algebra R , the set of the left-hand sides of all its identities $f = 0$ is a T -ideal in $F\langle X \rangle$, and it is well known that the set of varieties is in Galois correspondence with the set of T -ideals of $F\langle X \rangle$.

Let \mathcal{U} and \mathcal{V} be varieties of associative algebras. We denote by U and V , respectively, the corresponding T -ideals in $F\langle X \rangle$. The *product* \mathcal{UV} of the varieties \mathcal{U} and \mathcal{V} is defined by the T -ideal UV , i.e., an algebra R belongs to \mathcal{UV} iff it satisfies all the identities $uv = 0$, where $u \in U$ and $v \in V$.

Thus, Theorem 1.1 admits an obvious reformulation in term of products of varieties. Such a reformulation is a strict analog of the theorem on products of group varieties and of the theorem on products of the varieties of Lie algebras over an infinite field from [O] and [OS], respectively.

In turn, the papers [O] and [OS] extend the equality case $\mathcal{V}_1 \dots \mathcal{V}_k = \mathcal{V}'_1 \dots \mathcal{V}'_l$ considered earlier for group varieties in [3N] and [S], and for varieties of Lie algebras over an infinite field, in [P] and [B]. Indeed, " $=$ " is a special case of " \subseteq ", and the Neumanns–Shmel'kin theorem (the Parfenov–Bahturin theorem) says that the non-trivial group varieties (varieties of algebras Lie over an infinite field) form a free monoid under the product operation. (Recall that the product \mathcal{UV} of two group varieties contains all groups G having a normal subgroup N such that $N \in \mathcal{U}$ and $G/N \in \mathcal{V}$. The definition of the product for the varieties of Lie algebras is similar, although such a multiplication is not associative over finite fields [B1].)

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It might seem that the associative algebra case differs from the group and Lie algebra ones because the ideals of $F\langle X \rangle$ are usually not free. But instead of this, all right (and left) ideals are free modules over $F\langle X \rangle$, and making use of this fact, George Bergman and Jacques Lewin [BL] proved that the monoid of non-zero ideals of $F\langle X \rangle$ is free, and furthermore, the monoid of non-zero T -ideals of $F\langle X \rangle$ is also free.

The lemmas of the present papers are based on the Schreier–Lewin technique for firs (i.e., free ideal rings, see [C]) and on the technique of triangular products of modules invented by B.I. Plotkin (see [PG], [PV], [K], [V]). To complete the proof of Theorem 1.1, we follow the outline from [O].

The triangular products were originally introduced and applied to the products of varieties of group representations (see [PV], [V]). Let FG be the group algebra of a free group G with an infinite set of free generators $X = \{x_1, x_2, \dots\}$ over a field F , $f \in FG$ a 'polynomial', and H a group. Recall that an FH -module M is said to satisfy the identity $f = 0$ if for every $a \in M$ and every homomorphism $\mu : G \rightarrow H$, we have $a(\bar{\mu}(f)) = 0$, where $\bar{\mu}$ is the algebra homomorphism $FG \rightarrow FH$ induced by μ . A *variety of group representations* is the class of group representations (or modules over group algebras over F) satisfying a set of representation identities. A product \mathcal{UV} of such varieties is a variety consisting of all FG -modules M having a submodule N such that $N \in \mathcal{U}$ and $M/N \in \mathcal{V}$. This multiplication is obviously associative, and B.I. Plotkin [PG] proved that the monoid of non-zero varieties of group representations over any field F is free. The formulation and the proof of the following theorem are similar to those for 1.1.

Theorem 1.2. *Let \mathcal{V} and \mathcal{V}' be products of varieties of group representations over a field: $\mathcal{V} = \mathcal{V}_1 \dots \mathcal{V}_k$, and $\mathcal{V}' = \mathcal{V}'_1 \dots \mathcal{V}'_l$. Then we have $\mathcal{V} \subseteq \mathcal{V}'$ if and only if there exists a variety \mathcal{W} with two factorizations $\mathcal{W} = \mathcal{W}_1 \dots \mathcal{W}_k = \mathcal{W}'_1 \dots \mathcal{W}'_l$ such that $\mathcal{V}_1 \subseteq \mathcal{W}_1, \dots, \mathcal{V}_k \subseteq \mathcal{W}_k$ and $\mathcal{W}'_1 \subseteq \mathcal{V}'_1, \dots, \mathcal{W}'_l \subseteq \mathcal{V}'_l$.*

Theorem 1.2 solves a problem raised in [O].

2 Free module bases of T -ideals

We denote by X^* the free monoid generated by $X = \{x_1, x_2, \dots\}$. Recall that a subset $S \subseteq X^*$ is called a *Schreier set* if with any monomial (word) it contains all its left factors (prefixes).

Let I be a right ideal of $F\langle X \rangle$. Following Jacques Lewin [L], we call a set S of monomials a *Schreier basis* for $F\langle X \rangle$ modulo I if S is a Schreier set and the image of S in $F\langle X \rangle/I$ is an F -basis of the quotient $F\langle X \rangle/I$.

We denote by ϕ and ψ the endomorphisms of the algebra $F\langle X \rangle$ such that $\phi(x_i) = x_{i+1}$ for $i = 1, 2, \dots$ and $\psi(x_{i+1}) = x_i$ for $i = 1, 2, \dots$, $\psi(x_1) = 0$. Note that $\phi(I) \subseteq I$ and $\psi(I) \subseteq I$ if I is a T -ideal of $F\langle X \rangle$.

Lemma 2.1. *For any T -ideal I of $F\langle X \rangle$, there is a Schreier basis S_I of $F\langle X \rangle$ modulo I such that $\phi(S_I) \subseteq S_I$.*

Proof. Assume that the monoid X^* is well ordered by degrees and lexicographically for the monomials of the same degree. It is convenient to set $0 < u$ for every $u \in X^*$. Then for $u_1, u_2, u \in X^*$, we have $\phi(u_1) \leq \phi(u_2)$ iff $u_1 \leq u_2$, $\psi(u) < u$, and, provided $\psi(u_2) \neq 0$, we have $\psi(u_1) \leq \psi(u_2)$ if $u_1 \leq u_2$.

Call an element of X^* *I -reducible* if it lies in the sum of I and the span of the elements preceding it in the given ordering; call it *I -irreducible* otherwise. Then the images in $F\langle X \rangle/I$ of I -irreducible elements of X^* form a basis. Denote the set of I -irreducible elements of X^* by S_I .

That the set S_I is a Schreier set is equivalent to saying that if a left factor of a word is I -reducible, so is the whole word; but this is immediate (and only uses the definition of the well-ordering and the fact that I is a right ideal).

To see that $\phi(S_I) \subseteq S_I$, consider any $u \in S_I$ such that $\phi(u)$ is reducible. Thus, $\phi(u) - r \in I$ where r is an F -linear combination of monomials $< \phi(u)$. Applying ψ to this relation, we see that I contains $u - \psi(r)$, so u is also reducible; so as the contrapositive, we have that if u is irreducible, so is $\phi(u)$. \square

Lemma 2.2. *Every T -ideal I of $F\langle X \rangle$ is a free (right) $F\langle X \rangle$ -module having an $F\langle X \rangle$ -basis B such that $\phi(B) \subseteq B$.*

Proof. It is well-known that $F\langle X \rangle$ is a free right (and left) ideal ring or *right (and left) fir*, i.e., every right ideal I of $F\langle X \rangle$ is a free $F\langle X \rangle$ -module. Furthermore, all non-zero differences of the form $e_\alpha = sx_i - \sum \lambda_j s_j$ form an $F\langle X \rangle$ -basis B of the right module I ([C], Theorem VI.6.8). Here s is an arbitrary monomial from the Schreier F -basis S_I of $F\langle X \rangle$ modulo I , x_i is any element of X , and $\sum \lambda_j s_j$ is the unique linear combination of vectors s_j from S_I such that $sx_i = \sum \lambda_j s_j \pmod{I}$.

If $e_\alpha \neq 0$, then $\phi(e_\alpha) \in I \setminus \{0\}$ since ϕ is injective and I is a T -ideal. Then $\phi(e_\alpha) = \phi(s)x_{i+1} - \sum \lambda_j \phi(s_j)$, where $\phi(s), \phi(s_j) \in S_I$ by Lemma 2.1. Hence $\phi(e_\alpha)$ has the same form and therefore belongs to the basis B . The lemma is proved. \square

The following is a well known property of firs.

Lemma 2.3. *Let U, I and J be ideals of $F\langle X \rangle$ and $U \neq \{0\}$. Then $UI \subseteq UJ$ iff $I \subseteq J$.*

Proof. The ideal U has a non-empty basis (e_α) as right $F\langle X \rangle$ -module. Therefore a sum $\sum e_\alpha f_\alpha$ belongs to the product UI (to UJ) iff all the polynomials f_α belong to I (to J). This proves the statement of the lemma. \square

3 Annihilators and separators

Let M be a right $F\langle X \rangle$ -module and N a submodule. For a subset $Z \subseteq M$, its annihilator modulo N is

$$(N : Z) = \{a \mid a \in F\langle X \rangle, za \in N \text{ for all } z \in Z\}$$

The annihilator $(N : Z)$ is an ideal of $F\langle X \rangle$ if Z is a submodule of M .

Lemma 3.1. *If I and J are T -ideals of $F\langle X \rangle$, then $(I : J)$ is also a T -ideal in $F\langle X \rangle$.*

Proof. We must show that $\mu(a) \in (I : J)$ for every $a \in (I : J)$ and every endomorphism μ of $F\langle X \rangle$. Let $z = z(x_1, \dots, x_n) \in J$ and $a = a(x_1, \dots, x_n)$. Since J is a T -ideal, the polynomial $z(x_{n+1}, \dots, x_{2n})$ also belongs to J , and therefore $z(x_{n+1}, \dots, x_{2n})a(x_1, \dots, x_n) \in I$. Now we note that there is an endomorphism η of $F\langle X \rangle$ such that $\eta(x_1) = \mu(x_1), \dots, \eta(x_n) = \mu(x_n)$, $\eta(x_{n+1}) = x_1, \dots, \eta(x_{2n}) = x_n$. Since I is a T -ideal, we have

$$\eta(z(x_{n+1}, \dots, x_{2n})a(x_1, \dots, x_n)) = z(x_1, \dots, x_n)a(\mu(x_1), \dots, \mu(x_n)) = z\mu(a) \in I.$$

Since z is an arbitrary element of J , the lemma is proved. \square

Let V and U be ideals of $F\langle X \rangle$ and $V \subseteq U$. We introduce the concept of the *separator* $V \div U$ as follows.

There exists a unique smallest T -ideal L of $F\langle X \rangle$ such that $V \subseteq UL$. (This is an immediate corollary of the fir-property of $F\langle X \rangle$. Indeed, we can assume that $U \neq \{0\}$. Let $(e_\omega)_{\omega \in \Omega}$ be an $F\langle X \rangle$ -basis of the free right ideal U , and L' be a T -ideal of $F\langle X \rangle$. Then an element $\sum e_\omega f_\omega$ belongs to the product UL' iff all the coefficients f_ω belong to L' . Therefore $UL' \cap UL'' \cap \dots = U(L' \cap L'' \cap \dots)$ for arbitrary T -ideals L', L'', \dots) We denote this T -ideal L by $V \div U$.

Remark 3.2. There also exists a least ideal J such that $V \subseteq UJ$. However J is not necessarily a T -ideal even if both U and V are T -ideals of $F\langle X \rangle$. Let for example, U and V be minimal T -ideals of $F\langle X \rangle$ containing $\sum_{\pi \in S_3} \text{sign}(\pi)x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}$ and $\sum_{\pi \in S_4} \text{sign}(\pi)x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}x_{\pi(4)}$, respectively. Then it is easy to see that the ideal J consists of all polynomials with zero constant term. But J is not a T -ideal since it is not invariant under the endomorphism $x_1 \rightarrow 1, x_2 \rightarrow 0, x_3 \rightarrow 0, \dots$ of $F\langle X \rangle$.

Lemma 3.3. *Let I, J and U be T -ideals of $F\langle X \rangle$. Assume that $J \subseteq U$. Then*

$$(UI : J) = (I : (J \div U)).$$

Proof. We may assume that $U \neq \{0\}$. By Lemma 2.3, for arbitrary ideals V_1, V_2 and W of $F\langle X \rangle$, we have $UV_1W \subseteq UV_2$ iff $V_1W \subseteq V_2$. It follows that $(I : (J \div U)) = (UI : U(J \div U)) \subseteq (UI : J)$ because $J \subseteq U(J \div U)$. It remains to prove that $(UI : J) \subseteq (I : (J \div U))$.

Let $a = a(x_1, \dots, x_n) \in (UI : J)$, and so $za \in UI$ for every $z \in J$. With respect to the basis $B = (e_\alpha)$ of the (right) ideal U , given by Lemma 2.2, we have $z = \sum e_\alpha f_{z,\alpha}$, where $f_{z,\alpha} \in F\langle X \rangle$. Hence $f_{z,\alpha}a \in I$ for every $f_{z,\alpha}$.

The ideal $L = J \div U$ is the minimal T -ideal of $F\langle X \rangle$ containing all the polynomials $f_{z,\alpha} = f_{z,\alpha}(x_1, \dots, x_n)$. To complete the proof, it suffices to show that for every endomorphism $\mu : F\langle X \rangle \rightarrow F\langle X \rangle$ and every polynomial $g = g(x_1, \dots, x_n)$, we have $(\mu(f_{z,\alpha})g)a \in I$.

Recall that $\phi(x_i) = x_{i+1}$. Since J is a T -ideal, we have $\phi^n(z)g = \sum \phi^n(e_\alpha)\phi^n(f_{z,\alpha})g \in J$, and this is a basis decomposition of $\phi^n(z)g$ by Lemma 2.2. Therefore $\phi^n(f_{z,\alpha})ga \in I$, that is, $f_{z,\alpha}(x_{n+1}, \dots, x_{2n})g(x_1, \dots, x_n) \in I$. Since I is a T -ideal, it is invariant under an endomorphism of $F\langle X \rangle$ such that $x_1 \mapsto x_1, \dots, x_n \mapsto x_n, x_{n+1} \mapsto \mu(x_1), \dots, x_{2n} \mapsto \mu(x_n)$. Hence $f_{z,\alpha}(\mu(x_1), \dots, \mu(x_n))g(x_1, \dots, x_n)a(x_1, \dots, x_n) \in I$, as required. \square

For the next lemma, we need the construction of triangular product of modules invented by B.I. Plotkin and U.E. Kal'julaid ([PV],[K]). Note that every R -module P over an F -algebra R is a module over \bar{R} , the image of R in $\text{End}_F(P)$.

Let R_1 and R_2 be two F -algebras, M_1 a (right) R_1 -module, and M_2 an R_2 -module. Then the algebra $Q = \bar{R}_1 \oplus \bar{R}_2$ canonically becomes a subalgebra of $\text{End}_F(M_1 \oplus M_2)$. Let Φ consists of the F -linear operators on $M_1 \oplus M_2$ mapping M_1 to $\{0\}$ and M_2 to M_1 . Then $R = Q + \Phi$ is a subalgebra of $\text{End}_F(M_1 \oplus M_2)$, and so the vector space $M = M_1 + M_2$ is a faithful R -module. It is called the *triangular product* $M_1 \nabla M_2$ of the modules M_1 and M_2 . (Observe that M_1 is an R -submodule, but M_2 is not.)

Denote by $T(M_1)$ the T -ideal of $F\langle X \rangle$ consisting of all polynomials f vanishing under every homomorphism $F\langle X \rangle \rightarrow \bar{R}_1$. Similarly, one defines $T(M_2)$ and $T(M_1 \nabla M_2)$. Then $T(M_1 \nabla M_2) = T(M_2)T(M_1)$. (See [K] and Appendix in [V]. Of course, the inclusion \supseteq follows from the definitions.)

Lemma 3.4. *Let I, J , and K be T -ideals of $F\langle X \rangle$ and $K \not\subseteq I$. Then*

$$(IJ : K) = (I : K)J.$$

Proof. The inclusion \supseteq is obvious. To obtain \subseteq , we consider the right $F\langle X \rangle$ -modules $M_1 = F\langle X \rangle/J$, $M_2 = F\langle X \rangle/I$, and their triangular product $M = M_1 \nabla M_2$ which is an R -module as in the definition of triangular product. Since I and J are T -ideals, we have $T(M_1) = J$, $T(M_2) = I$, and therefore $T(M) = T(M_2)T(M_1) = IJ$. Also observe that $T(M_2K) = (I : K)$ by Lemma 3.1.

For a T -ideal L of $F\langle X \rangle$, we denote by L_R the corresponding T -ideal of R , that is the minimal ideal of R containing $\mu(L)$ for each of the homomorphisms $\mu : F\langle X \rangle \rightarrow R$. Note that L_R is not only generated as an ideal, but in fact is spanned as an F -vector-space by all $\mu(f)$, where $f = f(x_1, \dots, x_n) \in L$. Indeed, for arbitrary $r_1, r_2 \in R$ the product $r_1\mu(f)r_2$ is the image of the polynomial $x_{n+1}fx_{n+2}$ under a homomorphism η such that $\eta(x_1) = \mu(x_1), \dots, \eta(x_n) = \mu(x_n), \eta(x_{n+1}) = r_1$, and $\eta(x_{n+2}) = r_2$.

Since $K \not\subseteq I$, we have $M_2K \neq 0$, and so $N = MK_R \not\subseteq M_1$. Now, using F -linear operators from Φ , we observe that $N \supseteq M_1$. Furthermore, N regarded as \bar{R} -module, where \bar{R} is the image of R in $\text{End}(M_1 + M_2K)$, is isomorphic to $M_1 \nabla (M_2K)$. Therefore $T(N) = T(M_2K)T(M_1) = (I : K)J$. To complete the proof, it suffices to show that $T(N) \supseteq (IJ : K)$.

Assume that $a = a(x_1, \dots, x_n) \in (IJ : K)$, that is, $fa \in IJ$ for every $f = f(x_1, \dots, x_n) \in K$. Since K is a T -ideal, we also have $\bar{f}a \in IJ$, where $\bar{f} = f(x_{n+1}, \dots, x_{2n})$. Recall that $T(M) = IJ$, and so we have $v(\mu(\bar{f})\mu(a)) = 0$ for every $v \in M$ and every homomorphism $\mu : F\langle X \rangle \rightarrow R$. Since there are no variables x_i involved in both \bar{f} and a , we also have $(v(\mu_1(\bar{f}))\mu_2(a)) = 0$ for any two endomorphisms $\mu_1, \mu_2 : F\langle X \rangle \rightarrow R$, and so $(MK_R)\mu_2(a) = 0$ by the definition of K_R because we chose arbitrary $f \in K$. But $MK_R = N$, and therefore $a \in T(N)$, as desired. \square

4 Proofs of the theorems

Proof of Theorem 1.1. The remainder of the proof of Theorem 1.1 is similar to that in [O] and [OS]. The

'if' part of the statement is obvious, and we prove the 'only if' part below.

If $V' = \{0\}$, we can choose $I_1 = \dots = I_k = 0$ and $I'_1 = V'_1, \dots, I'_l = V'_l$. Hence we may suppose that $V' \neq \{0\}$ and induct on the number l of the factors in the decomposition of V' .

If $V \supseteq V'_1$, then there is an easy solution, namely, $I_1 = V_1, \dots, I_{k-1} = V_{k-1}, I_k = V_k V'_2 \dots V'_l$ and $I'_1 = V, I'_2 = V'_2, \dots, I'_l = V'_l$. So we may assume that $V \not\supseteq V'_1$, and consequently, $l \geq 2$.

Thus, there is j ($1 \leq j \leq k$) such that $V_1 \dots V_{j-1} \supseteq V'_1$ but $V_1 \dots V_{j-1} V_j \not\supseteq V'_1$. The T -ideal $U = V_1 \dots V_{j-1}$ is not $\{0\}$ since $U \supseteq V'_1 \supseteq V' \neq \{0\}$. Now the separator $V'_1 \div U = W_1$ is a T -ideal of $F\langle X \rangle$, and by Lemma 3.3, we get $(UV_j : V'_1) = (V_j : W_1)$. The ideal $W_2 = (UV_j : V'_1) = (V_j : W_1)$ is a T -ideal by Lemma 3.1. Recall that $W_1(V_j : W_1) \subseteq V_j$ by definition, whence

$$W_1 W_2 \subseteq V_j \quad (4.1)$$

Since $U = V_1 \dots V_{j-1}$, it follows from the definition of W_1 that

$$V_1 \dots V_{j-1} W_1 \supseteq V'_1 \quad (4.2)$$

Now we use Lemma 3.4 with $I = V_1 \dots V_j$, $J = V_{j+1} \dots V_k$, $K = V'_1$, whence

$$(V : V'_1) = W_2 V_{j+1} \dots V_k \quad (4.3)$$

by the definition of W_2 . It follows from the inclusion $V \supseteq V' = V'_1(V'_2 \dots V'_l)$ that $V'_2 \dots V'_l \subseteq (V : V'_1)$, and therefore by (4.3), we have

$$W_2 V_{j+1} \dots V_k \supseteq V'_2 \dots V'_l \quad (4.4)$$

The inclusion (4.4) and the inductive hypothesis imply the existence of a T -ideal L of $F\langle X \rangle$ with two T -ideal factorizations

$$L = L_j L_{j+1} \dots L_k = L'_2 \dots L'_l \quad (4.5)$$

such that

$$L_j \subseteq W_2, L_{j+1} \subseteq V_{j+1}, \dots, L_k \subseteq V_k; \quad V'_2 \subseteq L'_2, \dots, V'_l \subseteq L'_l \quad (4.6)$$

To complete the proof, we set

$$I_1 = V_1, \dots, I_{j-1} = V_{j-1}, I_j = W_1 L_j, I_{j+1} = L_{j+1}, \dots, I_k = L_k, \quad (4.7)$$

and

$$I'_1 = V_1 \dots V_{j-1} W_1, I'_2 = L'_2, \dots, I'_l = L'_l \quad (4.8)$$

Then it follows from (4.5–4.8) that $I_1 \dots I_k = I'_1 \dots I'_l (= I)$. Also $I_1 \subseteq V_1, \dots, I_{j-1} \subseteq V_{j-1}, I_{j+1} \subseteq V_{j+1}, \dots, I_k \subseteq V_k$ by (4.7) and (4.6), and $I_j = W_1 L_j \subseteq W_1 W_2 \subseteq V_j$ by (4.6) and (4.1). Finally, $V'_1 \subseteq I'_1$ by (4.8) and (4.2), and $V'_2 \subseteq I'_2, \dots, V'_l \subseteq I'_l$ by (4.8) and (4.6). The theorem is proved.

Remark 4.1. As G.Bergman and J.Lewin proved for $F\langle X \rangle$ that both the monoids of non-zero ideals and of non-zero T -ideals are free, it would be interesting to determine if one can omit all T -s in the formulations of Lemma 3.4 and Theorem 1.1 and replace the infinite set X by a finite one with $\text{card}(X) \geq 2$. Also it would be interesting to see to what extent the results can be generalized to more general firms.

Proof of Theorem 1.2. Let G be the free group freely generated by $X = \{x_1, x_2, \dots\}$ and FG the group algebra of G over a field F . Then the set of varieties of group representations over F is in Galois correspondence with the set of fully invariant ideals of FG (see [PV], Theorem I.2.1.2 or Section I.1 in [V]). By definition, these ideals are invariant under the endomorphisms of the free group G . Therefore it suffices to prove the analog of Theorem 1.1 for fully invariant ideals of FG . To obtain such a proof, we replace $F\langle X \rangle$ by FG , T -ideals by fully invariant ideals, and make the following minor alternation of our argument.

It is easy to see that 1 and all the products $u = (1 - x_{i_1}^{n_1}) \dots (1 - x_{i_t}^{n_t})$, where $t \geq 1, n_j \in \mathbb{Z}, n_j \neq 0, i_j \neq i_{j+1}$, form an F -basis X^* of FG . Now we take 'monomials' u of this form in the definition of Schreier set, replacing the word "prefix" by "left factor". By definition, $\deg u = |n_1| + \dots + |n_t|$. We suppose that X^* is well ordered by degrees and lexicographically if the degrees of two monomials are equal ($1 < 1 - x_1 < 1 - x_1^{-1} < 1 - x_2 < \dots$).

If we now set $\psi(x_1) = 1$ in the definition of the endomorphism ψ (and $\psi(x_{i+1}) = x_i$ for $i \geq 1$, as earlier), then the proofs of lemmas 2.1 and 2.2 work for every fully invariant ideal I of FG . The claim of Lemma 2.3 is also true since FG is a fir by [L].

There are no changes in the proof of the analogs of lemmas 3.1 and 3.3. In the definition of the triangular product, now R_1 and R_2 are group algebras of groups H_1 and H_2 , \bar{H}_1 and \bar{H}_2 are the canonical images of H_1 and H_2 in the F -linear groups $GL(M_1)$ and $GL(M_2)$, and so $Q = \bar{H}_1 \times \bar{H}_2$ is a subgroup of $GL(M_1 \oplus M_2)$, Φ is the group of F -linear operators of $M_1 \oplus M_2$ identical on the F -spaces M_1 and $(M_1 \oplus M_2)/M_1$, and $R = FH$, where $H = Q\Phi$. In the definition of $T(M_1)$, only those homomorphisms of group algebras are involved that are induced by group homomorphisms $G \rightarrow H_1$. The equality $T(M_1 \nabla M_2) = T(M_1)T(M_2)$ is Vovsi's theorem ([V], I.6.2). We must choose $r_1, r_2 \in H$ in the proof of the analog of Lemma 3.4. Then the proof of Theorem 1.1 just turns into the proof of Theorem 1.2.

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